Homework 4

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1. Consider the partial differential equation $u_t = u_x$, with initial data $u(x,0) = \phi(x)$. Solve it approximately as follows: put a grid on the (xt) plane, with mesh length h in the x-direction and k in the t-direction. Set $u_i^0 = \phi(ih)$. To calculate $u_{(i+1/2)h}^{(n+1/2)k}$ (half way between mesh points and half way up the time interval k) proceed as follows: pick a random number θ from the equidistibution density, one such choice for the whole half step. Set $u_{(i+1/2)h}^{(n+1/2)k} = u_i^n$ if $\theta \le 1/2 - k/(2h)$, $= u_{i+1}^n$ otherwise. The half-step from time $(n+1/2)k$ to $(n+1)k$ is similar. Show that if $k/h \le 1$ the solution of this scheme converges to the solution of the differential equation as $h \to 0$ (This is a special case of the Glimm or random choice scheme). Hint : The solution of the differential equation is $\phi(x+t)$, i.e., initial values propagate along the lines $t = -x + \mathit{constant}$. Examine how the scheme propagates initial values: show that an initial value u_i^0 moves in a time t by an amount η, where η is a random variable whose mean tends to $-t$ and whose variance tends to 0.

2. Consider the heat equation $v_t = (1/2)v_{xx}$ with initial data $v(x,0) = \phi(x)$ for $0 \le x \le 1$ and boundary conditions $v(0,t) = a$ and $v(1,t) = b$. Show that the solution at the point, $v(x,t)$, can be obtained by starting BM's from (x,t) backward in time, attaching to each BM a number F as follows : If the BM first hits the portion of the x-axis between the boundaries $x = 0, x = 1$, then $F = \phi(x + w(\omega, t))$; if it first hits the boundary $x = 0$ then $F = a$, and similarly at $x = 1$; finally, $v(x,t) = E[F]$.

Hint: One way is to go through a finite-difference argument, and then assume that the random walks converge to a BM. ,

3. Evaluate exactly $\int F dW$ for the following functionals F: (i)$F[w(\cdot)] = \int_0^1 w^4(s)ds$; (ii)$F = \sin(w^3(1))$.

4. Write the solution of the partial differential equation $v_t = (1/2)v_{xx} - xv$, with data $v(x,0) = \sin x$, as a Wiener integral.

5. Evaluate $\int F dW$, where $F[w(\cdot)] = e^{-\int_0^1 w^2(s)ds}\cos(w(1))$ by Monte-Carlo, as follows: Divide the time interval $[0,1]$ into n pieces; Construct random walks w_n as follows: For t a multiple of $1/n$, set $w_n((i+1)h) = w_n(ih) + q$, where q is a Gaussian variable with mean 0 and variance $1/n$ (

and of course $w_n(0) = 0$). For t between the multiples of $1/n$ construct w_n by linear interpolation. For each such w_n evaluate F and average over many walks, until the error (as measured by the difference between runs and also by a numerical estimate of the ds) is less than 1 per cent. Do this for $n = 5$ and $n = 10$. Note that this integral is the solution at $(0, 1)$ of some initial value problem for a differential equation. What is this problem?